GEOMETRY OF A SIMULTANEOUS SYSTEM OF TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY

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In a former paper† I have laid the foundation for a general theory of invariants of a system of linear homogeneous differential equations. I have actually determined the invariants for the special case of two equations, each of the second order, i. e., for the system

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

the independent variable being x. The transformations, which were considered, were the most general which could convert (1) into another system of the same form and order, viz: the transformations of the group G

$$(2)\qquad x = f(\xi), \quad y = \alpha(\xi)\eta + \beta(\xi)\zeta, \quad z = \gamma(\xi)\eta + \delta(\xi)\zeta,$$

where f, α, β, γ, δ are arbitrary functions of ξ, subject only to the condition that

$$\alpha\delta - \beta\gamma$$

must not vanish identically.

The present paper, besides deducing some new theorems, will be mainly concerned with geometrical interpretations. We shall again confine ourselves to the special case of equations (1) for two reasons. In the first place this will enable us to make use of the concrete results of our former paper, and in the second place we can thus avoid the consideration of configurations in hyperspace. It will not be difficult to generalize our considerations so as to include the general case, if only a space of the proper number of dimensions be employed.

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Geometrical considerations of a similar nature have been applied to the theory of a single linear differential equation by HALPHEN and FANO.*

§ 1. *Definition of the general solutions, and of a simultaneous fundamental system of solutions.*

According to the fundamental theorem of the theory of differential equations, the equations (1) define two functions of x, which are analytic if the coefficients are analytic, and which can be made to satisfy the further conditions that y, z, dy/dx, dz/dx assume arbitrarily prescribed values for the particular value of $x = x_0$, provided that the coefficients p_{ik}, q_{ik}, are holomorphic in the vicinity of $x = x_0$.

Such a system of functions, y and z, is said to constitute a system of general solutions of (1).

Now let (y_i, z_i) for $i = 1, 2, 3, 4$, be four systems of simultaneous solutions of the given system (1), so that

$$
(3) \qquad \begin{aligned} y_i'' + p_{11}y_i' + p_{12}z_i' + q_{11}y_i + q_{12}z_i = 0, \\ z_i'' + p_{21}y_i' + p_{22}z_i' + q_{21}y_i + q_{22}z_i = 0 \end{aligned} \qquad (i=1, 2, 3, 4).
$$

Then, denoting by c_1, c_2, c_3, c_4 four arbitrary constants,

$$
(4) \qquad y = \sum_{i=1}^{4} c_i y_i, \qquad z = \sum_{i=1}^{4} c_i z_i,
$$

will also form a simultaneous system of solutions. Moreover, they constitute a system of general solutions, if, and only if, the determinant

$$
(5) \qquad D = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix}
$$

does not vanish identically. For then, if x_0 is a value of x for which $D \neq 0$, $c_1 \cdots c_4$ can be so determined as to make y, z, dy/dx, dz/dx equal to arbitrarily assigned constants for $x = x_0$.

If (y_i, z_i) are such functions of x which satisfy (1) and do not make D identically zero, we say of them that they constitute a *fundamental system* of simultaneous solutions. Equations (4) then furnish the general integrals of system (1).

We can express the condition

$$
D \neq 0
$$

* For the history of this subject Mr. FANO's excellent paper in Mathematische Annalen, vol. 53, may be consulted.

in another way. If $D = 0$, it is possible to find four functions of x, λ, μ, ν, ρ, so that the four equations

$$\lambda y_k + \mu y_k' + \nu z_k + \rho z_k' = 0 \qquad (k = 1, 2, 3, 4), \tag{6}$$

may be verified.

If (y_k, z_k) form a fundamental system of simultaneous solutions, it must therefore be impossible to find functions λ, μ, ν, ρ so as to satisfy (6), or what amounts to the same thing, it must be impossible to find functions α, β, γ, δ which satisfy the system of equations

$$\begin{aligned} \alpha y_1 + \beta y_2 + \gamma y_3 + \delta y_4 &= 0, \\ \alpha y_1' + \beta y_2' + \gamma y_3' + \delta y_4' &= 0, \\ \alpha z_1 + \beta z_2 + \gamma z_3 + \delta z_4 &= 0, \\ \alpha z_1' + \beta z_2' + \gamma z_3' + \delta z_4' &= 0, \end{aligned} \tag{7}$$

which, in particular, prevents $y_1 \cdots y_4$ and $z_1 \cdots z_4$ from verifying simultaneous relations of the form

$$\sum_{i=1}^{4} c_i y_i = 0, \quad \sum_{i=1}^{4} c_i z_i = 0, \tag{8}$$

where $c_1 \cdots c_4$ are constants.

Suppose now that four pairs of functions (y_i, z_i) which verify no relations of the form (6) or (7) are given. Then we can consider them as constituting a fundamental system of simultaneous solutions of a system of differential equations of the form (1). The relations between the coefficients p_{ik}, q_{ik} and the functions (y_k, z_k) follow from (3). They are

$$\begin{aligned} Dp_{11} &= -D(y_k'', z_k', y_k, z_k), & Dp_{12} &= -D(y_k', y_k'', y_k, z_k), \\ Dp_{21} &= -D(z_k'', z_k', y_k, z_k), & Dp_{22} &= -D(y_k', z_k'', y_k, z_k), \\ Dq_{11} &= -D(y_k', z_k', y_k'', z_k), & Dq_{12} &= -D(y_k', z_k', y_k, y_k''), \\ Dq_{21} &= -D(y_k', z_k', z_k'', z_k), & Dq_{22} &= -D(y_k', z_k', y_k, z_k''), \end{aligned} \tag{9}$$
$$D = D(y_k', z_k', y_k, z_k),$$

where we have adopted the notation

$$D(a_k, b_k, c_k, d_k) = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \tag{10}$$

We have moreover

$$p_{11} + p_{22} = -\frac{1}{D}\frac{dD}{dx}, \tag{11}$$

whence

$$D = Ce^{-\int (p_{11}+p_{22})dx}, \tag{12}$$

where C is a non-vanishing constant.

All of these theorems are simple generalizations of well-known facts, but it is necessary for the purposes of this paper to formulate them explicitly.

If we subject the general solutions of our system (1) to a transformation of the form

$$\eta = \alpha y + \beta z, \quad \zeta = \gamma y + \delta z, \tag{13}$$

where α, β, γ, δ are functions of x, then η and ζ will be the general solutions of a system of differential equations of the same form as (1). Moreover, if we put

$$\begin{aligned} \eta_i &= \alpha y_i + \beta z_i, \\ \zeta_i &= \gamma y_i + \delta z_i \end{aligned} \qquad (i=1, 2, 3, 4), \tag{13a}$$

the four pairs of functions (η_i, ζ_i) will form a definite fundamental system of the new system of differential equations, and its general solutions will be

$$\eta = \sum_{i=1}^{4} c_i \eta_i, \quad \zeta = \sum_{i=1}^{4} c_i \zeta_i.$$

Thus, if we consider instead of a pair of general solutions of system (1), four pairs of solutions which form a fundamental system, these are transformed cogrediently with each other, and with the pair of general solutions.

§ 2. *Geometrical interpretation.*

Let us interpret (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) as the homogeneous coördinates of two points in space. If (1) is integrated, we shall have all of these quantities expressed as functions of x:

$$y_k = f_k(x), \quad z_k = g_k(x) \qquad (k=1, 2, 3, 4), \tag{14}$$

so that we can say that the system (1) defines two curves in space C_y and C_z, whose points moreover are put into a definite correspondence with one another, namely, those being corresponding points that belong to the same value of x.

But there is a restriction on these curves, owing to the condition that (y_i, z_i) are to be the members of a fundamental system, so that equations (6) must not be verified. Let us write (6) as follows

$$\lambda y_k + \mu y'_k = -(\nu z_k + \rho z'_k) \qquad (k=1, 2, 3, 4). \tag{6a}$$

Now the quantities $\lambda y_k + \mu y'_k$ for each value of x denote the homogeneous coördinates of some point on the tangent to the curve C_y constructed at the point whose parameter is x, and $\nu z_k + \rho z'_k$ denote the coördinates of a point on the tangent to C_z at the corresponding point of this curve. If equation (6a) is satisfied these two tangents intersect for all pairs of corresponding points.

In order, then, that the curves C_y and C_z may be the integral curves of a system of form (1), *it is necessary and sufficient that the tangents of these curves at all corresponding points do not intersect.*

In particular the curves may be plane curves, but they must not be in the same plane.

What is the geometrical meaning of transformations of form (13)?

Let us mark on the curves C_y and C_z the points P_y and P_z corresponding to the same value of x, and let us join them by a straight line L_{yz}. Then it is clear that the transformations (13), or more properly (13a), convert the points P_y and P_z of the line L_{yz} into two other points P_η and P_ζ of the same line. Moreover, since α, β, γ, δ are arbitrary functions of x, it becomes possible to convert P_y and P_z into *any* other two points of this line. The line L_{yz} then is invariant for all such transformations.

If we make this construction for all values of x, we obtain a ruled surface S, whose generators are the straight lines L_{yz}. A change of the independent variable

$$x = f(\xi),$$

where $f(\xi)$ is an arbitrary function, interchanges the generators in the most general way.

Thus, there belongs to every system of two linear homogeneous differential equations of the second order a ruled surface, which we shall call the integrating ruled surface, whose generators are the lines joining corresponding points of the two integral curves. This ruled surface is the same for all such systems which can be transformed into each other by a transformation of the form

$$\eta = \alpha y + \beta z, \qquad \zeta = \gamma y + \delta z, \qquad \xi = f(x), \tag{15}$$

where α, β, γ, δ, f are arbitrary functions of x.

There is one important restriction, however, viz.: *the ruled surface cannot be a developable surface.* For, in the case of a developable surface, and only in that case, would the corresponding tangents of the two integral curves be coplanar, and therefore intersect.

This consideration also teaches us the meaning of the singular values of the independent variable for which

$$D = 0.$$

They are the values of x for which two consecutive generators of the ruled surface intersect, i. e., they correspond to the generators upon which are situated singular points of the surface.

It should be remarked that the ruled surface S will be different for different fundamental systems of solutions of (1), if we regard the tetrahedron of reference as fixed. All of the ruled surfaces belonging to the different possible fundamental systems of the same system of differential equations are obtained from one of them by projective transformation.

By means of equations (13a) we associated with each fundamental system of the original system of equations, a definite fundamental system of the transformed system of equations, and it is only for such associated fundamental systems of the two systems of equations that it is true that they have the same integrating ruled surface. In general the two integrating ruled surfaces will only be projective transformations of each other. If we call two systems of differential equations of form (1) *equivalent*, when they can be transformed into each other by a transformation of the form (15), we can state our theorem more precisely as follows:

If two systems of differential equations of form (1) *are equivalent, their integrating ruled surfaces are projective transformations of each other. Moreover if the fundamental systems of solutions be properly selected, the ruled surfaces coincide.*

Conversely, if the ruled surfaces of two such systems coincide, the systems are equivalent.

If the ruled surface is not of the second order, this converse is clear at once. For the arbitrary functions α, β, γ, δ in the transformation (13) can be chosen so as to convert any pair of curves on the surface, which are not generating straight lines, into any other pair, for instance the pair of curves corresponding to the first system into that corresponding to the second. Moreover this ensures the equivalence of the two systems, if the surface is not of the second order. But if it is of the second order it may be generated by straight lines in two different ways. But even then the theorem holds, since we can transform one set of generators into the other by a projective transformation.

Suppose that any non-developable ruled surface S is given. There corresponds to it a class of mutually equivalent systems of linear differential equations. To find a representative of this class, we trace any two curves, not generating straight lines, on the surface and express the coördinates of their points as functions of a parameter x, in such a way that to the same value of x correspond points of the two curves which are situated upon the same generator of the surface. The system of differential equations whose integral curves are these, parametrically represented in this manner, is the required representative system.

We see further that it is always possible, by a transformation of form (15) to convert any system of differential equations of the kind here considered into another with plane integral curves. We can even take the planes of these curves parallel, and moreover this can be done in an infinity of ways.

If the ruled surface is algebraic, the system can be transformed into one whose integral curves are algebraic space or plane curves.

Any equation or system of equations between p_{ik}, q_{ik}, *etc., which remains invariant for all of the transformations of the form* (15), *expresses a projective property of the integrating ruled surface.* For, it expresses a property common to all possible pairs of curves on this surface, and p_{ik}, q_{ik}, p'_{ik}, $\cdots$ are differential invariants of the general projective group of space. *Conversely any projective property of the integrating ruled surface can be expressed by an invariant system of equations.*

§ 3. *Expression of the integrating ruled surface in line coördinates.*

The surface S is generated by the motion of the line L_{yz} which joins the points $y = (y_1, y_2, y_3, y_4)$ of the curve C_y and $z = (z_1, z_2, z_3, z_4)$ of the curve C_z.

The Plückerian line-coördinates of L_{yz} are the six determinants of the second order in the matrix

$$\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

Put

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, 2, 3, 4), \tag{16}$$

then the identical relation will hold

$$\omega_{12}\omega_{34} + \omega_{23}\omega_{14} + \omega_{31}\omega_{24} = 0. \tag{17}$$

Corresponding to the transformations of the infinite group G, we shall have for ω_{ik} the transformations

$$\omega_{ik} = (\alpha\delta - \beta\gamma)\bar{\omega}_{ik}, \quad x = f(\xi).$$

But the six line-coördinates will verify a linear homogeneous differential equation, in general of the 6th order, say

$$P_0\omega^{(6)} + P_1\omega^{(5)} + P_2\omega^{(4)} + P_3\omega^{(3)} + P_4\omega^{(2)} + P_5\omega^{(1)} + P_6\omega = 0. \tag{18}$$

Now the most general transformations which convert such an equation into another of the same form and order are those of the infinite group H,

$$\omega = \phi(x)\bar{\omega}, \quad x = f(\xi),$$

where f and ϕ are arbitrary functions. Such functions of $P_1, \cdots, P_6$, $P'_1, \cdots, P'_6$, etc., which remain unaltered by all transformations of the group H are called invariants of (18).

It is clear, then, that the invariants of the linear differential equation of the sixth order, which the line-coördinates of a generator of the integrating ruled *surface satisfy, are also invariants of the original system of differential equations.*

We proceed to set up this linear differential equation of the sixth order in a normal form. We have shown, in a former paper, that every system of form (1) can be reduced to a particular form, which we have called the *semi-canonical* form, for which $p_{ik} = 0$.* Let (y, z) and (η, ζ) be any two simultaneous solutions of a system of differential equations in the semi-canonical form, so that we shall have

$$(19) \qquad \begin{aligned} y'' &= -q_{11}y - q_{12}z, \qquad & \eta'' &= -q_{11}\eta - q_{12}\zeta, \\ z'' &= -q_{21}y - q_{22}z, \qquad & \zeta'' &= -q_{21}\eta - q_{22}\zeta. \end{aligned}$$

Let us put

$$(20) \qquad \omega = y\zeta - z\eta = (y\zeta).$$

We wish to find the differential equation satisfied by ω. To do this, let us first notice the relations

$$(21) \quad (\eta y'') = +q_{12}\omega, \quad (\eta z'') = +q_{22}\omega, \quad (\zeta y'') = -q_{11}\omega, \quad (\zeta z'') = -q_{21}\omega,$$

which are obtained from (19). Then, by successive differentiation, and using the abbreviations

$$(22) \qquad \begin{aligned} 2v &= \omega'' + (q_{11} + q_{22})\omega, \\ w &= v'' - 2(q_{11}q_{22} - q_{12}q_{21})\omega + (q_{11} + q_{22})v, \\ t &= w' - (q_{11}q'_{22} + q_{22}q'_{11} - q_{12}q'_{21} - q_{21}q'_{12})\omega + (q'_{11} + q'_{22})v, \\ \rho &= t' - (q_{11}q''_{22} + q_{22}q''_{11} - q_{12}q''_{21} - q_{21}q''_{12})\omega + (q''_{11} + q''_{22})v, \end{aligned}$$

we obtain the system of equations

$$(23) \qquad \begin{aligned} \omega' &= -(\eta z') + (\zeta y'), \\ v' &= q_{11}(\eta z') + q_{12}(\zeta z') - q_{21}(\eta y') - q_{22}(\zeta y'), \\ w &= q'_{11}(\eta z') + q'_{12}(\zeta z') - q'_{21}(\eta y') - q'_{22}(\zeta y'), \\ t &= q''_{11}(\eta z') + q''_{12}(\zeta z') - q''_{21}(\eta y') - q''_{22}(\zeta y'), \\ \rho &= q^{(3)}_{11}(\eta z') + q^{(3)}_{12}(\zeta z') - q^{(3)}_{21}(\eta y') - q^{(3)}_{22}(\zeta y'). \end{aligned}$$

Eliminating the determinants $(\eta z')$, etc., we find the required differential equation of the 6th order for ω, viz:

$$(24) \qquad \begin{vmatrix} \rho, & q^{(3)}_{11}, & q^{(3)}_{12}, & -q^{(3)}_{21}, & -q^{(3)}_{22} \\ t, & q''_{11}, & q''_{12}, & -q''_{21}, & -q''_{22} \\ w, & q'_{11}, & q'_{12}, & -q'_{21}, & -q'_{22} \\ v', & q_{11}, & q_{12}, & -q_{21}, & -q_{22} \\ \omega', & -1, & 0, & 0, & +1 \end{vmatrix} = 0.$$

*Transactions of the American Mathematical Society, vol. 2 (1901), no. 1, p. 19.

This differential equation reduces to the 5th order if the minor of ρ in this determinant is zero, i. e., if

$$\begin{vmatrix} q''_{11} - q''_{22}, & q''_{12}, & q''_{21} \\ q'_{11} - q'_{22}, & q'_{12}, & q'_{21} \\ q_{11} - q_{22}, & q_{12}, & q_{21} \end{vmatrix} = 0,$$

or, when the equation is not written in the semi-canonical form, if

$$(25) \qquad \Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix} = 0.$$

It is clear from this, that Δ must be an invariant, obviously of weight nine. This may also be verified analytically. Δ, being isobaric of weight 9, cannot be rationally expressed as a function of the seminvariants involved in it, for it can be shown that no such rational invariant of weight 9 exists. But there is a simple syzygy between Δ and the invariant of weight 18.* By expressing both of these quantities in terms of u_{ik}, v_{ik}, w_{ik}, it will be found that

$$(26) \qquad \Delta^2 + 4\theta_{18} = 0.$$

We have seen that the differential equation for ω reduces to the 5th order if and only if $\Delta = 0$.

Therefore, the equation

$$(27) \qquad \Delta = 0$$

means, that the generators of the integrating ruled surface belong to a linear complex; for, the six line coördinates ω_{ik} must, in this case, satisfy a homogeneous linear relation with constant coefficients. The condition $\Delta = 0$ is necessary and sufficient.

If the linear complex, to which the generators of the ruled surface belong, is special, additional relations must be fulfilled. For any two linear complexes which are not special can be transformed into each other by a projective transformation, while a non-special complex can never be so transformed into a special complex.

A special complex consists of all lines which intersect a given straight line, called the axis of the complex. If

$$\omega_{12}a_{34} + \omega_{23}a_{14} + \omega_{31}a_{24} + \omega_{14}a_{23} + \omega_{24}a_{31} + \omega_{34}a_{12} = 0$$

is its equation, then a_{ik} are the line-coördinates of its axis, and

$$a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24} = 0.$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 18.

Every ruled surface contained in such a complex, therefore has upon it a straight line which is not a generator of the surface, namely the axis of the complex. Take this as our fundamental curve C_y, and also as one of the edges of the tetrahedron of reference.

Then we can put

$$y_3 = y_4 = 0 \tag{28}$$

while y_1, y_2; z_1, z_2, z_3, z_4 may be arbitrary functions of x. Since $y_3 = y_4 = 0$ we can take as the first differential equation of our system the linear differential equation of the second order of which y_1, y_2 form a fundamental system, so that

$$p_{12} = q_{12} = 0,$$

whence follows

$$u_{12} = v_{12} = w_{12} = 0, \tag{29}$$

which makes

$$\theta_{10} = \theta_{15} = \theta_{18} = 0, \tag{30}$$

while θ_4 and θ_6 do not vanish, unless there is a further specialization. Since, moreover, in this case

$$\theta_4 = (u_{11} - u_{22})^2,$$

we may assume

$$u_{11} - u_{22} \neq 0. \tag{29a}$$

The conditions (29) and (29a) are sufficient to insure that the integrating ruled surface shall be generated by lines belonging to a special linear complex, for from them follows $p_{12} = q_{12} = 0$. But they are not necessary. The necessary and sufficient conditions must form an invariant system of equations, which the conditions here found do not do. The conditions (30) on the other hand are necessary, as they form an invariant system, but they are not sufficient.

To find conditions for this case, which are both necessary and sufficient we proceed as follows. If the ruled surface belongs to a special linear complex, it must be possible to transform the given system, by a transformation of the form

$$y = a_{11}\eta + a_{12}\zeta,$$

$$z = a_{21}\eta + a_{22}\zeta,$$

into another, for which $\pi_{12} = \rho_{12} = 0$, if the coefficients of the transformed system be denoted by Greek letters.

But the conditions $\pi_{12} = \rho_{12} = 0$, give

$$2(a'_{12}a_{22} - a'_{22}a_{12}) + p_{11}a_{12}a_{22} + p_{12}a_{22}^2 - p_{21}a_{12}^2 - p_{22}a_{22}a_{12} = 0, \tag{31}$$

and

$$\begin{aligned} a''_{12}a_{22} - a''_{22}a_{12} + p_{11}a'_{12}a_{22} + p_{12}a'_{22}a_{22} - p_{21}a'_{12}a_{12} - p_{22}a'_{22}a_{12} \\ + q_{11}a_{12}a_{22} + q_{12}a_{22}^2 - q_{21}a_{12}^2 - q_{22}a_{22}a_{12} = 0. \end{aligned} \tag{32}$$

Differentiating the left member of (31) and subtracting from the resulting expression twice the left member of (32), we find

$$(33)\quad \begin{aligned} &(a_{12}a'_{22} - a'_{12}a_{22})(p_{11} + p_{22}) + a_{12}a_{22}[p'_{11} - 2q_{11} - (p'_{22} - 2q_{22})] \\ &\qquad + a^2_{22}(p'_{12} - 2q_{12}) - a^2_{12}(p'_{21} - 2q_{21}) = 0. \end{aligned}$$

Multiplying (31) by $-(p_{11} + p_{22})$ and (33) by 2, and adding we find

$$(34)\qquad a_{12}a_{22}(u_{11} - u_{22}) + a^2_{22}u_{12} - a^2_{12}u_{21} = 0,$$

where the quantities u_{ik} are the same as those so denoted in our previous paper.

Dividing (34) by a^2_{22}, we have

$$\frac{a_{12}}{a_{22}}(u_{11} - u_{22}) + u_{12} - \left(\frac{a_{12}}{a_{22}}\right)^2 u_{21} = 0,$$

whence by differentiation

$$(35)\quad \left(u_{11} - u_{22} - 2\frac{a_{12}}{a_{22}}u_{21}\right)\frac{d}{dx}\frac{a_{12}}{a_{22}} = -u'_{12} - \frac{a_{12}}{a_{22}}(u'_{11} - u'_{22}) + \left(\frac{a_{12}}{a_{22}}\right)^2 u'_{21}.$$

But from (31) we have

$$2\frac{d}{dx}\frac{a_{12}}{a_{22}} = -\frac{a_{12}}{a_{22}}(p_{11} - p_{22}) - p_{12} + \left(\frac{a_{12}}{a_{22}}\right)^2 p_{21},$$

which gives on substitution in (35), after simplifying and clearing of fractions,

$$(36)\qquad a_{12}a_{22}(v_{11} - v_{22}) + a^2_{22}v_{12} - a^2_{12}v_{21} = 0.$$

Differentiating (36), we get by a repetition of this process

$$(37)\qquad a_{12}a_{22}(w_{11} - w_{22}) + a^2_{22}w_{12} - a^2_{12}w_{21} = 0.$$

We have then three equations (34), (36) and (37) for

$$a = \frac{a_{12}}{a_{22}}.$$

It is at once seen that Δ must vanish, and if we put

$$(38)\quad \begin{aligned} &2(v_{12}w_{21} - v_{21}w_{12}) = U, \quad 2(w_{12}u_{21} - w_{21}u_{12}) = V, \quad 2(u_{12}v_{21} - u_{21}v_{12}) = W, \\ &vw_{12} - wv_{12} = U_{12}, \qquad wu_{12} - uw_{12} = V_{12}, \qquad uv_{12} - vu_{12} = W_{12}, \\ &vw_{21} - wv_{21} = U_{21}, \qquad wu_{21} - uw_{21} = V_{21}, \qquad uv_{21} - vu_{21} = W_{21}, \\ &\qquad u_{11} - u_{22} = u, \quad v_{11} - v_{22} = v, \quad w_{11} - w_{22} = w, \end{aligned}$$

we obtain

$$(39)\quad \begin{aligned} a &= -\frac{U}{2U_{21}} = -\frac{V}{2V_{21}} = -\frac{W}{2W_{21}}, \\ a^2 &= \frac{U_{12}}{U_{21}} = \frac{V_{12}}{V_{21}} = \frac{W_{12}}{W_{21}}, \end{aligned}$$

whence

$$
(40)\qquad \begin{aligned}
VW_{21} - WV_{21} &= WU_{21} - UW_{21} &&= UV_{21} - VU_{21} = 0,\\
V_{12}W_{21} - W_{12}V_{21} &= W_{12}U_{21} - U_{12}W_{21} &&= U_{12}V_{21} - V_{12}U_{21} = 0,\\
U^2 - 4U_{12}U_{21} &= V^2 - 4V_{12}V_{21} &&= W^2 - 4W_{12}W_{21} = 0.
\end{aligned}
$$

It will be noted that U, V, W, etc., are the nine minors of the second order of the determinant Δ, and that *all of the minors of the second order of their determinant*

$$
(41)\qquad \begin{vmatrix} U & U_{12} & U_{21} \\ V & V_{12} & V_{21} \\ W & W_{12} & W_{21} \end{vmatrix}
$$

vanish in consequence of the relations (40). Moreover the quantities U, U_{12}, U_{21}; V, V_{12}, V_{21}; W, W_{12}, W_{21}; are all cogredient with each other and u, u_{12}, u_{21}, etc., for transformations of the dependent variables, and we can write

$$
(42)\qquad \theta_{10} = \tfrac{1}{4}W^2 - W_{12}W_{21}.
$$

The conditions (40) which are necessary for the case under consideration are also sufficient. For, let us suppose these conditions fulfilled for a system of differential equations. Then let us transform this system as before, putting now

$$
(43)\qquad a_{12} = -\tfrac{1}{2}W, \quad a_{22} = W_{21}.
$$

We find by actual computation that the left member of (31), which is equal to π_{12}, vanishes. Denoting this left member, in general, by π_{12}, and the left member of (32) by ρ_{12}, the left member of (34) will be

$$
2\pi'_{12} - 4\rho_{12} - (p_{11} + p_{22})\pi_{12}.
$$

But this vanishes under the given conditions, and since $\pi_{12} = 0$, ρ_{12} also will vanish. Therefore the first equation of the transformed system will actually have the form

$$
(44)\qquad \frac{d^2\eta}{dx^2} + \pi_{11}\frac{d\eta}{dx} + \rho_{11}\eta = 0,
$$

which proves that the ruled surface in this case belongs to a special linear complex. This proof is insufficient only if $W = W_{21} = 0$.

If $W = W_{21} = 0$, we have either $u_{21} = v_{21} = 0$, or else also $W_{12} = 0$. In the first case, if $u_{21} = v_{21} = 0$, and $u_{11} - u_{22} \neq 0$ we find from the equations defining these quantities that p_{21} and q_{21} must vanish, which means that in this case no transformation is necessary, one of the original equations already being of form (44). If however, in addition to $u_{21} = v_{21} = 0$, we have $u = 0$, we find from (40) that either $W_{12} = 0$, which comes under our second case, or else U, V, U_{21} and

V_{21} must all vanish. But this again gives either $u=0$, $u_{12}=0$, $u_{21}=0$, which we shall see later makes the ruled surface of the second order, or else $w_{21}=0$. This again gives rise to two sub-cases. Either $p_{21}=q_{21}=0$ as in a former case, or $v=0$ and therefore $u_{12}=0$, which is again the case of a surface of the second order. In all of these cases the ruled surface belongs to a special linear complex.

In the second case mentioned above, we have $W=W_{12}=W_{21}=0$, or

$$u_{12}v_{21}-u_{21}v_{12}=0, \quad uv_{12}-vu_{12}=0, \quad uv_{21}-vu_{21}=0,$$

whence, denoting by ρ a proportionality factor,

$$v=\rho u, \quad v_{12}=\rho u_{12}, \quad v_{21}=\rho u_{21}, \tag{45}$$

unless $u=u_{12}=u_{21}=0$, which is the case of a surface of the second degree.

Using the equations (32) of our former paper,* which defined v_{ik} we find from (45)

$$\begin{aligned}\rho u &= 2u'+2(p_{12}u_{21}-p_{21}u_{12}),\\ \rho u_{12} &= 2u'_{12}+(p_{11}-p_{22})u_{12}-p_{12}u,\\ \rho u_{21} &= 2u'_{21}-(p_{11}-p_{22})u_{21}+p_{21}u,\end{aligned}$$

and similarly, from the equations (39) in our former paper,

$$w=(\rho^2+2\rho')u, \quad w_{12}=(\rho^2+2\rho')u_{12}, \quad w_{21}=(\rho^2+2\rho')u_{21}, \tag{46}$$

but these expressions of v_{ik}, w_{ik} show that, in this case, not only

$$W=W_{12}=W_{21}=0,$$

but also

$$U=U_{12}=U_{21}=V=V_{12}=V_{21}=0,$$

i. e., all of the minors of the second order of Δ vanish. In this case also, the ruled surface belongs to a special linear complex although our method of reduction fails. That this is so will, however, be apparent from what follows.

We have seen, then, that the equations (40) *are the necessary and sufficient conditions for an integrating ruled surface belonging to a special linear complex.* Moreover we have seen how the axis of this complex, i. e., the straight line on the surface, which is not one of the generators, may be found, for this is the geometrical meaning of the transformation which we have just completed. The cases in which this transformation fails are those in which more than one such straight line exists on the surface.

If the generators of the ruled surface belong to two different linear complexes,

$$\begin{aligned}\Omega_1 &= \omega_{12}a_{34}+\omega_{23}a_{14}+\omega_{31}a_{24}+\omega_{14}a_{23}+\omega_{24}a_{31}+\omega_{34}a_{12}=0,\\ \Omega_2 &= \omega_{12}b_{34}+\omega_{23}b_{14}+\omega_{31}b_{24}+\omega_{14}b_{23}+\omega_{24}b_{31}+\omega_{34}b_{12}=0,\end{aligned} \tag{47}$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 8.

or if they are lines of what PLÜCKER calls a linear congruence, they belong to every one of the ∞^1 complexes

$$\Omega_1 + \lambda\Omega_2 = 0. \tag{48}$$

Now, among these there are in general two, which are special. For (48) represents a special linear complex, if and only if,

$$B\lambda^2 + C\lambda + A = 0, \tag{49}$$

where we have put

$$\begin{aligned} A &= a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24}, \\ B &= b_{12}b_{34} + b_{23}b_{14} + b_{31}b_{24}, \\ C &= a_{12}b_{34} + a_{23}b_{14} + a_{31}b_{24} + a_{34}b_{12} + a_{14}b_{23} + a_{24}b_{31}. \end{aligned} \tag{50}$$

But (49) has two distinct and determinate roots if

$$C^2 - 4AB \neq 0.$$

In this case we can assume that the complexes (47) are special, so that $A = B = 0$, and consequently $C \neq 0$. The congruence consists of all of the lines intersecting two distinct, non-intersecting straight lines, the directrices of the congruence.

If $C^2 - 4AB = 0$, we have a congruence with coincident directrices.

A sub-case of this,

$$A = B = C = 0,$$

must be excluded from our investigation. For in this case the directrices of the congruence intersect, so that the congruence consists of all straight lines intersecting a pair of intersecting straight lines. In other words, all of the lines of the congruence are situated either in the plane of the directrices, or else pass through their point of intersection. Any ruled surface contained in such a congruence is either a plane or a cone, and therefore a developable surface. But a developable surface cannot be the integrating ruled surface of our system of differential equations.

It will be easily seen that the necessary and sufficient conditions, which must be fulfilled in order that the generators of the integrating ruled surface may belong to a linear congruence, are that all of the minors of the second order of Δ must vanish.

In this case then

$$U = U_{12} = U_{21} = V = V_{12} = V_{21} = W = W_{12} = W_{21} = 0, \tag{51}$$

and as a consequence,

$$\theta_{10} = \theta_{15} = \theta_{18} = 0.$$

In the case of a congruence with coincident directrices additional conditions must be fulfilled. Such a congruence may be described in a number of ways. It consists of all of the lines of a complex which intersect a given line of the complex. Or, we may consider the line with which the two coincident directrices of the congruence coincide, as a generator of a surface of the second order. All of the tangents to this surface, passing through the given generator, constitute the lines of the congruence. Finally, for actual use, we can define such a congruence in the following manner. Take rectangular axes x, y, z in space. Through a point P on the x axis, and the origin O pass a plane, whose projection on the yz plane makes an angle λ with the z axis, where

$$\tan \lambda = \frac{k}{x},$$

k being a constant, and x the distance of P from the origin. Then the lines of this plane, which pass through P, are the lines of the congruence.*

Let us take the x axis, which must be a line on our surface, intersected by all of the generators, as the fundamental curve C_y, and the abscissa $OP = x$ of the point P as the independent variable of our system of differential equations. Moreover, suppose that we have chosen our homogeneous coördinates so that the ratios

$$\frac{y_1}{y_4}, \quad \frac{y_2}{y_4}, \quad \frac{y_3}{y_4}; \qquad \frac{z_1}{z_4}, \quad \frac{z_2}{z_4}, \quad \frac{z_3}{z_4};$$

are the cartesian coördinates of the points (y) and (z) respectively. Then we can put

$$(52) \qquad y_1 = x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1.$$

Moreover the equation of the plane mentioned above, will be

$$y = z \tan \lambda,$$

whence follows the single relation between z_2 and z_3,

$$z_2 = z_3 \tan \lambda = z_3 \frac{k}{x},$$

or

$$(53) \qquad xz_2 - kz_3 = 0.$$

There exists then, in this case, a system of differential equations, equivalent to the given system, for which

$$(54) \qquad \begin{aligned} y_1 &= x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1, \\ z_1 &= z_1, \quad z_2 = z_2, \quad z_3 = \frac{x}{k} z_2, \quad z_4 = 1, \end{aligned}$$

* PLÜCKER, *Neue Geometrie des Raumes*, pp. 57 and 73.

where z_1 and z_2 are arbitrary functions of x.

Substituting these values in equations (9), we find

$$
(55)\qquad
\begin{aligned}
&p_{11} = p_{12} = q_{11} = q_{12} = 0, \\
&p_{21} = -\frac{1}{z_2^2}\left[z_1''z_2^2 - 2z_2z_1'z_2' - (x - z_1)\{2(z_2')^2 - z_2z_2''\}\right], \\
&p_{22} = -2\frac{z_2'}{z_2}, \\
&q_{21} = -\frac{1}{z_2^2}\left[2(z_2')^2 - z_2z_2''\right], \quad q_{22} = +\frac{1}{z_2^2}\left[2(z_2')^2 - z_2z_2''\right],
\end{aligned}
$$

whence we derive the relations

$$
(56)\qquad
\begin{aligned}
&2p_{22}' - 4q_{22} + p_{22}^2 = 0, \\
&q_{21} + q_{22} = 0, \\
&p_{11} = p_{12} = q_{11} = q_{12} = 0.
\end{aligned}
$$

Therefore

$$
(57)\qquad u_{11} = u_{12} = u_{22} = 0, \quad v_{11} = v_{12} = v_{22} = 0, \quad w_{11} = w_{12} = w_{22} = 0.
$$

Substituting these values, it will be seen that *all of the invariants vanish.*

The conditions (57) are sufficient to make the ruled surface belong to a congruence with coincident directrices. For, if we assume in the first place the additional relation $u_{21} = 0$, the surface will be of the second order, as we shall show very soon, and therefore it will be a surface of the kind here considered. If $u_{21} \neq 0$, we find from the conditions $u_{11} = u_{12} = v_{11} = 0$, that p_{12} must vanish. From $u_{12} = 0$ then follows $q_{12} = 0$, and from $u_{11} = u_{22} = 0$,

$$
(58)\qquad 2p_{11}' - 4q_{11} + p_{11}^2 = 2p_{22}' - 4q_{22} + p_{22}^2 = 0.
$$

Our system therefore has the form

$$
\begin{aligned}
&y'' + p_{11}y' + q_{11}y = 0, \\
&z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0,
\end{aligned}
$$

with the relations (58). If we now transform, by putting

$$
y = e^{-\frac{1}{2}\int p_{11}dx}\eta, \qquad z = e^{-\frac{1}{2}\int p_{22}dx}\zeta,
$$

the system will become, owing to the relations (58),

$$
\begin{aligned}
&\eta'' = 0, \\
&\zeta'' + \pi\eta' + \kappa\eta = 0,
\end{aligned}
$$

where π and κ are functions of x subject to no condition.

But this system has the fundamental system of solutions

$$\eta_1 = x, \qquad \eta_2 = 0, \quad \eta_3 = 0, \quad \eta_4 = 1,$$
$$\zeta_1 = -\int\int(\pi + \kappa x)\,dx^2, \quad \zeta_2 = k, \quad \zeta_3 = x, \quad \zeta_4 = -\int\int \kappa dx^2,$$

where k is a constant, which must be different from zero, for we find that

$$D(\eta_k', \zeta_k', \eta_k, \zeta_k) = k.$$

Making another transformation, by putting

$$\bar{\zeta} = \frac{\zeta}{-\int\int \kappa dx^2},$$

it will be seen that $\eta_1, \eta_2, \eta_3, \eta_4$ and $\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3, \bar{\zeta}_4$ form a fundamental system of solutions of a system equivalent to the given system, and of the form (54). This proves that any system of differential equations, for which the relations (57) are satisfied, has a ruled surface which belongs to a linear congruence with coincident directrices.

But the sufficient conditions (57) are not necessary. The following conditions

(59) $$I^2 - 4J = 0, \quad K - I'^2 = 0, \quad L - 4I''^2 = 0,$$

are both necessary and sufficient. For, in the first place it may be verified that they form an invariant system of equations.

In the second place, any system of differential equations of the kind here considered can be so transformed that the relations (57) shall be satisfied. But then the relations (59) will also be satisfied, and moreover, since these relations form an invariant system of equations, they must have been satisfied for the untransformed system of differential equations. The equations (59) are then necessary conditions. But they are also sufficient. For, suppose that they are satisfied. We can then assume

$$u_{11} - u_{22} = 0, \quad u_{12}u_{21} = 0.$$

For, we can, by a transformation of the form $y = \lambda\eta$, $z = \mu\zeta$, make u_{11} and u_{22} equal to any functions of x whatever, and therefore it may be so chosen as to make $u_{11} = 0$ and $u_{22} = 0$.

Moreover, since then $u_{12}u_{21} = 0$, either u_{12} or u_{21} must vanish. Suppose $u_{21} \neq 0$; then $u_{12} = 0$, and $v_{12} = 0$, while

$$v_{11} - v_{22} = 2p_{12}u_{21}$$

must vanish, on account of (59). But since $u_{21} \neq 0$, we have $p_{12} = 0$, whence $v_{11} = v_{22} = 0$, and also $w_{11} = w_{22} = w_{12} = 0$, or, taken together, the sufficient conditions (57). If $u_{21} = 0$, together with $u_{11} = u_{22} = u_{12} = 0$, we have the case of a ruled surface of the second order, which also belongs to a linear congruence with coincident directrices. Had we assumed $u_{12} \neq 0$, instead of $u_{21} \neq 0$, we should have found the conditions

$$u_{11} = u_{21} = u_{22} = 0, \quad v_{11} = v_{21} = v_{22} = 0, \quad w_{11} = w_{21} = w_{22} = 0,$$

which are the same as (57) except that the order of the two differential equations is interchanged, i. e., except for the notation.

The necessary and sufficient conditions for a ruled surface belonging to a linear congruence with coincident directrices, are therefore

$$(60) \qquad \begin{aligned} (u_{11} - u_{22})^2 + 4u_{12}u_{21} &= 0, \\ (v_{11} - v_{22})^2 + 4v_{12}v_{21} &= 0, \\ (w_{11} - w_{22})^2 + 4w_{12}w_{21} &= 0, \end{aligned}$$

for these are equivalent to (59).

If the ruled surface is made up of the lines common to three distinct linear complexes, it is a surface of the second order (not a cone), and all of the elements of the determinant Δ must vanish.

Therefore the necessary and sufficient conditions for an integrating ruled surface of the second order are

$$(61) \qquad u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

This is the case treated in our former paper in § 4. It is possible to so transform the system that the new dependent variables η and ζ satisfy the same linear differential equation of the second order

$$\eta'' + p\eta' + q\eta = 0;$$

the geometrical sense of this reduction is that any two generators of the second kind are taken as the fundamental curves C_y and C_z. The differential equations for y and z are then the same because the point-rows cut out of any two generators of the second kind by a moving generator of the surface are projective.

This reduction can be made analytically by reducing to the semi-canonical form.

§ 4. *Geometrical interpretation of the semi-canonical form of the system of differential equations.*

We have shown in our former paper, that every system of differential equations of the form here considered, can be reduced to the form

$$(62) \qquad \begin{aligned} y'' + p_{11}y + p_{12}z &= 0, \\ z'' + p_{21}y + p_{22}z &= 0, \end{aligned}$$

which we have called the semi-canonical form. We have also determined the most general subgroup of the group (2), which leaves this semi-canonical form invariant. Its finite equations are,

$$(63)\qquad \xi = \xi(x), \quad \eta = \sqrt{\frac{d\xi}{dx}}\,(C_{11}y + C_{12}z), \quad \zeta = \sqrt{\frac{d\xi}{dx}}\,(C_{21}y + C_{22}z),$$

where C_{ik} are constants.

Now, of course, equations (62) define a pair of curves on the integrating ruled surface. This is transformed by equations (63) into another pair, whose defining system of differential equations also has the semi-canonical form.

Since (63) is the most general transformation which leaves the semi-canonical form invariant, *we see that there exists upon the integrating ruled surface a single infinity of curves, whose defining systems of differential equations have the semi-canonical form. It is also clear from equations* (63) *that a moving generator of the ruled surface intersects any four of these curves in a row of four points whose anharmonic ratio is constant.*

On any ruled surface the generating straight lines constitute one set of asymptotic lines. *We shall now prove that the curves just mentioned constitute the other set of asymptotic lines.*

Let the homogeneous coördinates be so chosen that

$$\frac{y_1}{y_4},\ \frac{y_2}{y_4},\ \frac{y_3}{y_4};\quad \frac{z_1}{z_4},\ \frac{z_2}{z_4},\ \frac{z_3}{z_4};$$

are the cartesian coördinates of the points describing the curves C_y and C_z respectively. Then if X, Y, Z denote the coördinates of any point in the osculating plane of the curve C_y at the point (y_1, y_2, y_3, y_4), the equation of this plane is

$$\begin{vmatrix} X - \frac{y_1}{y_4}, & Y - \frac{y_2}{y_4}, & Z - \frac{y_3}{y_4} \\ \frac{d}{dx}\frac{y_1}{y_4}, & \frac{d}{dx}\frac{y_2}{y_4}, & \frac{d}{dx}\frac{y_3}{y_4} \\ \frac{d^2}{dx^2}\frac{y_1}{y_4}, & \frac{d^2}{dx^2}\frac{y_2}{y_4}, & \frac{d^2}{dx^2}\frac{y_3}{y_4} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} Xy_4 - y_1, & Yy_4 - y_2, & Zy_4 - y_3 \\ y_1y_4' - y_4y_1', & y_2y_4' - y_4y_2', & y_3y_4' - y_4y_3' \\ y_1y_4'' - y_4y_1'', & y_2y_4'' - y_4y_2'', & y_3y_4'' - y_4y_3'' \end{vmatrix} = 0.$$

But since y_i, z_i are solutions of (62), we have

$$y_i'' = -p_{11}y_i - p_{12}z_i, \qquad z_i'' = -p_{21}y_i - p_{22}z_i \qquad (i=1, 2, 3, 4),$$

and, therefore,

$$y_k y_i'' - y_i y_k'' = -\rho_{12}(z_i y_k - y_i z_k),$$

which shows that the osculating plane of C_y at the point (y_1, y_2, y_3, y_4) also passes through (z_1, z_2, z_3, z_4), i. e., it coincides with the plane tangent to the surface at the point (y_1, y_2, y_3, y_4). This proves that C_y is an asymptotic curve of the surface.

Our remarks furnish a new proof of PAUL SERRET'S theorem: *the double ratio of the four points, in which any generator intersects four fixed asymptotic curves of the second kind, is constant.*

We shall leave the geometrical interpretation of the canonical form for a future occasion, when we shall also treat some other important problems not touched upon in this paper. In particular, it will be interesting to introduce the idea of duality into this theory.

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